

Explicit Formulas for the Scalar Modes in Seiberg–Witten Theory with an Application to the Argyres–Douglas Point

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ABSTRACT: General formulas for the scalar modes a_i and a_D^i in the Seiberg–Witten $SU(N)$ setting are derived in the cases with and without massive hypermultiplets. Subsequently these formulas are applied in a study of the $SU(3)$ Argyres–Douglas point. We use this example to study the question, whether the scalar modes admit an interpretation in terms of BPS mass states everywhere in moduli space. The paper collects, in an appendix, various facts on the function Lauricella $F_D^{(n)}$, which naturally appears in the derived formulas.

KEYWORDS: dgf, set, sud.

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1. Introduction

In a much celebrated work [25], Seiberg and Witten found an exact solution to $N = 2$ supersymmetric four-dimensional Yang–Mills theory with gauge group $SU(2)$. This paper initiated a whole new, tree-sized, branch of research leading to a vast set of exactly solvable Yang–Mills theories in various dimensions and with various degrees of supersymmetry. For some basic or introductory works see, for example, [27, 28, 16, 1, 15, 19, 8], and references therein. Of particular interest for these solutions is the understanding of the moduli space of vacua, which in many cases turns out to be a hyperelliptic Riemann surface. In particular, simply-laced Lie groups lead to spectral curves which are hyperelliptic.

The BPS spectrum of such a model is entirely determined by the periods of a special meromorphic differential 1-form on this Riemann surface, the Seiberg–Witten differential λ_{SW} . A general hyperelliptic Riemann surface Σ can be described in terms of two variables w, Z in the polynomial form

$$w^2 + 2A(Z)w + B(Z) = 0 \tag{1.1}$$

with $A(Z), B(Z) \in \mathbb{C}[Z]$. After a simple coordinate transformation in $y = w - A(Z)$, this takes on the more familiar form $y^2 = A(Z)^2 - B(Z)$. But we might also write the hyperelliptic curve in terms of a rational map if we divide the defining equation 1.1 by $A(Z)^2$ and put $\tilde{w} = w/A(Z) + 1$ to arrive at the representation

$$(1 - \tilde{w})(1 + \tilde{w}) = \frac{B(Z)}{A(Z)^2}. \quad (1.2)$$

This form is very appropriate in the frame of Seiberg–Witten models, since the Seiberg–Witten differential can be read off directly: The rational map $R(Z) = B(Z)/A(Z)^2$ is singular at the zeroes of $B(Z)$ and $A(Z)$, and is degenerate whenever its Wronskian $W(R) \equiv W(A(Z)^2, B(Z)) = (\partial_Z A(Z)^2)B(Z) - A(Z)^2(\partial_Z B(Z))$ vanishes. This is precisely the information encoded in λ_{SW} which for arbitrary hyperelliptic curves, given by a rational map $R(Z) = B(Z)/A(Z)^2$, can be expressed as

$$\lambda_{\text{SW}} = \frac{Z}{2\pi i} d(\log \frac{1 - \tilde{w}}{1 + \tilde{w}}) = \frac{1}{2\pi i} d(\log R(Z)) \frac{Z}{\tilde{w}} = \frac{1}{2\pi i} \frac{W(A(Z)^2, B(Z))}{A(Z)B(Z)} \frac{Z dZ}{y}. \quad (1.3)$$

Note that the denominator polynomial being a square guarantees the curve to be hyperelliptic. It is this local form of the Seiberg–Witten differential which serves as a metric $ds^2 = |\lambda_{\text{SW}}|^2$ on the Riemann surface. And it is this local form which arises as the tension of self-dual strings coming from 3-branes in type II string theory compactifications on Calabi–Yau threefolds.

The crucial point in Seiberg–Witten theory is that the physically relevant information of the BPS mass spectrum is encoded in the periods of the above defined 1-form. In general, the hyperelliptic curve has $2N_c$ branch points and is thus of genus $g = N_c - 1$. Any element γ of the homology of the curve defines a period and hence the mass of a BPS state,

$$Z_\gamma = \oint_\gamma \lambda_{\text{SW}}, \quad m_\gamma = |Z_\gamma|. \quad (1.4)$$

If N_f (massive) hypermultiplets are present, the homology basis gets extended by N_f loops encircling each of the differential’s poles in order to pick up residue terms, if the contour γ is deformed in such a way that it crosses a pole.

The appealing picture emerging from the analysis of the Seiberg–Witten solution of supersymmetric low-energy effective field theories is that the geometry of an auxiliary Riemann surface Σ entirely determines the physics. Given the surface and a special 1-form on it, the properties of particles are encoded in the periods of this particular 1-form, the Seiberg–Witten differential. Choosing a canonical homology basis $\{\alpha_i, \beta^i\}$ with $1 \leq i < N_c$, such that $\alpha_i \cap \alpha_j = \beta^i \cap \beta^j = 0$ and $\alpha_i \cap \beta^j = \delta_i^j$, defines electric-like and magnetic-like quantum numbers for particles. More precisely, every $\gamma \in H_1(\Sigma, \mathbb{Z})$ can be written as a linear combination with integer coefficients,

$$\gamma = \sum_i q^i \alpha_i + g_i \beta^i, \quad (1.5)$$

defining in turn dual pairs of quantum numbers (q^i, g_i) . Stable particles are those, for which all these pairs consist of coprime numbers. As we see, the definition of ‘electric’ and

‘magnetic’ quantum numbers is fixed by the choice of a basis in the homology of Σ . It is generically independent of small variations of the location of the branch points of the Riemann surface, i.e. the basis is generically invariant under small variations of the moduli of the surface. However, there is typically no single canonical homology basis, which can be used for all possible variations of the moduli. This will concern us in some detail later on. This is an important point, since varying the moduli usually implies a deformation of the contours, which encircle, once and for all, chosen pairs of branch points. However, since the Seiberg–Witten differential has zeroes and poles, such deformations are not always possible without crossing one of these.

The approach taken in this paper emphasizes the geometrical picture for the BPS states, namely that they are described in terms of period integrals taken along homology contours. In this picture, one immediately sees which BPS states can become massless under what circumstances. This is especially easy to see, when the Riemann surface is realized as a branched covering of the complex plane (or sphere). As soon as two branch points coalesce, the BPS state corresponding to the cycle γ_{ij} , which encloses precisely the two branch points e_i and e_j and none of the poles of the Seiberg–Witten differential, will become massless, since $Z_{\gamma_{ij}} \rightarrow 0$ for $\gamma_{ij} \rightarrow 0$, i.e. for $|e_i - e_j| \rightarrow 0$ under a suitable variation of the moduli. Another important advantage of our approach is that we do not need to find Picard–Fuchs equations, which is a very hard task for gauge groups of rank greater than two.

1.1 Outline of motivation and strategy

Up to here, we have not really deviated from the common point of view. However, we would like to emphasize the following point in this work: Even if it is clear that the periods $a_\gamma(u)$ of the Seiberg–Witten differential $\lambda_{\text{SW}}(u)$ are complex analytic functions of the moduli u_k of the Riemann surface Σ , it appears to us that the interpretation of a given $a_\gamma(u)$ as describing BPS states for all of the moduli space spanned by the u_k may break down. The reason is hidden in the complicated and highly non-trivial properties of the functions $a_\gamma(u)$ under analytic continuation. Let us start with a given $a_\gamma(u)$ for such values of the u_k that it is perfectly well defined as a period integral along the cycle γ . Then it has an asymptotic expansion in the u_k . Now we seek its analytic continuation for other patches in the moduli space of the u_k , which can be done, for example, by studying the monodromy properties of $a_\gamma(u)$. Of course, analytic continuations exist for any patch in the moduli space. But what strikes us is that many of these seem not to have an interpretation as simple period integrals anymore.

More precisely, we claim that the set of Seiberg–Witten periods, given in terms of a basis of functions $a_i = a_{\alpha_i}(u)$ and $a_D^j = a_{\beta_j}(u)$ for a suitable patch in the moduli space spanned by the u_k , does not close under monodromy or analytic continuation. This is to be contrasted with the well-known fact that, for example, the standard Gaussian hypergeometric system, spanned by two solutions of its defining second-order ordinary differential equation, is complete under analytic continuation. This is precisely, what happens in the simplest case of gauge group $SU(2)$. The resulting Riemann surface is elliptic, and the Seiberg–Witten periods can be recast in a form given in terms of ordinary hypergeometric

functions. Thus, the theory can be explored in all of moduli space, i.e. for all values of the single complex modulus u . As we will show, higher rank gauge groups, which lead to hyperelliptic surfaces, involve generalized hypergeometric functions of several variables. In fact, we will have to deal with the particular set of functions $F_D^{(n)}$, called Lauricella functions of type D . First of all, the defining differential equations are now partial differential equations, from which one might already guess that things get worse. Indeed, not much is known about analytic continuations of these functions. However, one thing is clear, namely that the set of the functions $F_D^{(n)}$ for a given n is not sufficient in order to define analytic continuations everywhere. The problem is that one has to add further functions to this set, a different sort of generalized hypergeometric functions $D_{p,q}^{(n)}$. These functions seem not to have an integral representation which can be interpreted as a period integral.

Thus, our analysis raises the question whether analytic continuations of Seiberg–Witten periods always possess an interpretation as central charges for BPS mass states. Unfortunately, the complete answer to this question is complicated by the fact that the map between the $N_c - 1$ moduli u_k , $k = 1, \dots, N_c - 1$, and Λ of the Riemann curve Σ and the $2N_c$ branch points e_i is multi-valued and rather opaque. A generic hyperelliptic surface of genus $g = N_c - 1$ has a moduli space of (complex) dimension $2g - 1 = 2N_c - 3$. This is clearly not exhausted by the N_c complex numbers u_k and Λ for $N_c > 3$. For this reason, will we concentrate mainly on the case of gauge group $SU(3)$, since then the dimensions of the moduli space and the space spanned by $u_1 \equiv u$, $u_2 \equiv v$ and Λ match. Thus, the coordinates u, v, Λ may serve *locally* as coordinates of the moduli space. For $N_c > 3$, the vacuum expectation values u_k and the cut-off Λ will only sweep out a physically relevant sub-variety of the full hyperelliptic moduli space. Our findings are nonetheless of a general nature and hold true also for $N_c > 3$. It is worth noting here that adding of N_f massive hypermultiplets, if we allow for complex masses m_r , can make up for the missing moduli. In fact, $N_c - 1$ vacuum expectation values, $N_f \leq N_c$ masses and the cut-off Λ precisely sum to $2N_c$ complex numbers. It is well known that the theory with $N_f = N_c$ massive hypermultiplets is conformally invariant, which reflects the fact that conformal transformations can change three of the $2N_c$ numbers to whatever we want¹, leaving $2N_c - 3$ free variables, which is precisely the dimension of the hyperelliptic moduli space.

As mentioned above, the case $SU(3)$ without hypermultiplets is the simplest case we can consider. The phenomenon we want to study is, however, generally valid. The Seiberg–Witten differential, expressed in the branch points of the Riemann surface Σ , has a very rich structure of singular sub-manifolds where special things may happen. The best known case is that of two of the branch points flowing into each other. This causes the associated BPS state to become massless. But as soon as the number of branch points exceeds four, the possibility arises that more than two of them might run into each other. In fact, we may consider $2g - 1$ of the $2g + 2$ branch points of a generic hyperelliptic surface to be

¹Actually, what can be transferred by conformal transformations is the location of the branch points, and three of these can be mapped to any desired values. However, the $2N_c$ branch points are functions of the $N \leq 2N_c$ numbers u_k , m_r and Λ . If $N = 2N_c$, we can locally invert these functions to obtain locally valid coordinates in moduli space.

free variables, fixing the remaining three to, e.g. 0, 1, and ∞ .² There are very many ways in which branch points can form several subsets where all points in a subset are close to coinciding with each other. Let us call the free variables x_i , $i = 1, \dots, 2g - 1$. Typically, one first calculates the Seiberg–Witten periods in a patch of moduli space where all x_i are small. This patch corresponds to the singular point $(0, 0, \dots, 0) \in \mathbb{C}^{2g-1}$. Now, the number M of singular sub-manifolds intersecting at this point is given by the $2g - 1$ possibilities $x_i = 0$ together with the $\binom{2g-1}{2}$ possibilities that $x_i = x_j$ for $i \neq j$. This makes in total $M = g(2g - 1)$ possibilities. We see that for gauge group $SU(2)$, i.e. $g = 1$, this simply yields $M = 1$ possibility, so nothing spectacular is expected to happen. But already for gauge group $SU(3)$, i.e. $g = 2$, we get $M = 6$ sub-manifolds, which intersect at the singular point $(0, 0, 0)$. The general case is given by singular points $(\underbrace{0, \dots, 0}_p, \underbrace{1, \dots, 1}_q, \underbrace{\infty, \dots, \infty}_{2g-1-p-q})$ or permutations thereof. The number M of intersecting sub-manifolds now is

$$M = \frac{1}{2}p(p + 1) + \frac{1}{2}q(q + 1) + \frac{1}{2}(2g - 1 - p - q)(2g - p - q). \quad (1.6)$$

Considerable difficulties arise when one attempts to obtain a *complete* set of solutions of any given hypergeometric partial differential system valid in an entire neighbourhood of such a singular point, as soon as more than $2g - 1$ singular sub-manifolds intersect, i.e. as soon as $M \geq 2g$. In fact, it appears that it is impossible to obtain a single fundamental set of solutions valid in the entire neighbourhood of such a singular point. Instead, one can only construct sets of solutions valid in hypercones whose common vertex is the singular point in question.

The hypergeometric system we will encounter here is the Lauricella system $F_D^{(n)}$. It is worth noting the peculiar property that it is impossible to construct any one fundamental set of solutions valid in the whole neighbourhood of any possible singular points by the methods known and employed here, whenever the number of free variables exceeds three. Thus, we have a chance to deal with $SU(3)$ without hypermultiplets using only the standard methods for analytic continuation. We will do so later, and obtain in this way solutions valid in entire neighbourhoods of certain singular points. Already the case $SU(4)$ is out of reach for a complete treatment of the neighbourhood of even one singular point.

We remark here, that our approach works with a completely factorized form of the Seiberg–Witten differential. Thus, our functions will formally depend on the branch points, the poles (given by the masses of the hypermultiplets and the point at infinity, where the Seiberg–Witten differential has a double pole) and the zeroes of λ_{SW} . Even though the zeroes are also functions of the moduli and masses and are thus not independent of the branch points and poles it is more convenient, and also more economic, to treat the zeroes as additional variables. It does not influence our argument later on, since the Seiberg–Witten period integrals can always be decomposed by linearity into integrals depending solely on the branch points and the poles. It is the number of truly independent variables which counts. For gauge group $SU(2)$, we have just one variable, and hence it is no surprise

²Which three we choose is up to us and can be adapted to the situation at hand. Thus, in the following discussion we can assume without loss of generality that the singular point in question is of the form $(0, \dots, 0, 1, \dots, 1, \infty, \dots, \infty)$.

that everything can be reduced to ordinary hypergeometric functions with their well-known analytic properties. For all higher rank gauge groups, we have at least two independent variables. The minimal case we can then run into is the Appell function F_1 , a generalized hypergeometric function of two variables. It is known that a complete set of solutions sufficing to construct analytical continuations in whole neighbourhoods around singular points consists out of 25 integrals, only ten of these being of type F_1 , the other 15 being Horn’s functions of type G_2 . The latter do not possess integral representations in terms of contour intergals along simple (Pochhammer) loops. In fact, double circuit loops are needed as well, where each of the two loops encircles two singular points. Thus, we arrive at the statement announced above that the analytical continuation, although existent, does not everywhere admit an interpretation in terms of periods of scalar modes yielding BPS mass states.

1.2 Structure of the paper

In the following, we will work out this line of argument in more detail, concentrating mainly on one particular example, namely the Argyres–Douglas \mathbb{Z}_3 -point for the case of gauge group $SU(3)$, since this is precisely such a singular point where several singular sub-manifolds intersect.

In order to carry out this analysis we derive general formulas for the scalar modes in Seiberg–Witten theory which involve the above-mentioned Lauricella $F_D^{(n)}$. These formulas work directly in the picture of a branched ramified covering of the complex plane and thus do not rely on the derivation and solutions of the associated system of Picard–Fuchs equations.

This paper is organized as follows: We begin by stating the necessary preliminaries from ‘Seiberg–Witten theory.’ Then we derive our formulas for the scalar modes with and without hypermultiplets for gauge group $SU(N)$. As an application we consider Argyres–Douglas’ \mathbb{Z}_3 -point. This is followed by a section “Conclusions and Outlook,” where we discuss some possible consequences of our findings, suggesting directions of further investigation.

The paper closes with a rather long appendix on the Lauricella functions $F_D^{(n)}$ which naturally appear in our derived formulas. The idea is to collect all necessary results on these functions in order to make the paper more self-contained.

2. General formulas for the scalar modes

In this section we derive our formulas for the scalar modes occuring in the BPS mass formula. As was remarked in the introduction, one of our devices is the factorization of the Seiberg–Witten differential so that the integrals involved in our calculations can be identified as Lauricella functions of type D .

In the first subsection we gather the relevant expression for the Seiberg–Witten differential and so on. Then, in the second subsection, we consider the $SU(N)$ case without massive hypermultiplets, whereas in the third subsection we consider the case with massive hypermultiplets.

For completeness, we mention that integration of the Seiberg–Witten form can be achieved by other methods. First of all, the traditional way is by setting up and solving the Picard–Fuchs system of the curve Σ with given form λ_{SW} . Another method integrates the Seiberg–Witten differential directly, after recasting it in modular coordinates with respect to the charge lattice generated by its periods [18]. Conformal invariance or isomonodromic properties of the periods can also be used to find them in explicit form [7, 12]. Finally, for certain cases such as $SU(2)$, functional properties of the prepotential related to a certain kind of modular invariance can also be employed [20, 21], see also [22] for an approach where the prepotential is computed directly, without determining the scalar modes first. Most of these methods become increasingly more difficult or cannot be applied at all for higher rank gauge groups.

2.1 The Seiberg–Witten differential

As is well known, the exact Seiberg–Witten solutions of SUSY Yang–Mills theories are obtained like this: One writes down a certain hyperelliptic curve,³ often called the spectral curve of the theory, and the BPS spectrum for the particular theory is then determined by the periods of a given meromorphic 1-form λ_{SW} on this curve.

For $N = 2$ supersymmetric $SU(N)$ Yang–Mills theory without massive hypermultiplets the curve-defining equation is (the N is the N in $SU(N)$)

$$y^2 = A(x)^2 - B := \left(x^N - \sum_{k=2}^N u_k x^{N-k} \right)^2 - \Lambda^{2N} = \prod_{i=1}^{2N} (x - e_i), \quad (2.1)$$

and the Seiberg–Witten differential is

$$\lambda_{\text{SW}} = \frac{1}{2\pi i} \frac{\prod_{\ell=0}^{N-1} (x - z_\ell)}{\prod_{i=1}^{2N} \sqrt{x - e_i}} dx, \quad (2.2)$$

where the z_ℓ ($\ell > 0$) are the zeros of $2A'(x)B$, $z_0 = 0$ and the e ’s are the zeros of y^2 . Usually the e ’s are called *branch points*.

This form of the Seiberg–Witten differential is not unique. We could add exact terms to it which would not affect the computation of period integrals. However, the form given here is singled out in string-theoretic derivations of Seiberg–Witten low-energy effective field theories. There, the spectral curve Σ appears through intersecting branes, and the Seiberg–Witten differential is induced by the metric on Σ which arises from the tension of self-dual strings (in a derivation from type II string theory).

With these data one proceeds as follows: Let $\{\alpha_i, \beta^i\}_{1 \leq i \leq N-1}$ denote a canonical homology basis for our curve, i.e. one for which $\alpha_i \cap \beta^j = \delta_i^j$, then the scalar modes are given by $a_i = \int_{\alpha_i} \lambda_{\text{SW}}$, $a_D^j = \int_{\beta^j} \lambda_{\text{SW}}$ and in terms of these quantities the mass of a BPS state with charge (\mathbf{q}, \mathbf{g}) is given by $m_{(\mathbf{q}, \mathbf{g})} \propto |q^i a_i + g_j a_D^j|$. In a string-theoretic derivation, the homology cycles arise as projections from three-cycles in Calabi–Yau compactifications.

³To be precise, there are cases where the curve is not hyperelliptic. This happens for gauge groups which are not simply-laced. However, we shall not consider those cases.

Other derivations of Seiberg–Witten theory exist, e.g. from M -theory, but they agree in the canonical choice of λ_{SW} and its periods.

The case with massive hypermultiplets brings about some changes in the spectral curve as well as in λ_{SW} which we shall communicate in due course (see Section 2.3 below).

In deriving our results, we will be making use of the following integral identity involving Lauricella $F_D^{(n)}$; this function is considered more fully in the appendix as well as in [10]:

$$\int_0^1 t^{a-1} (1-t)^{c-a-1} \prod_{i=1}^n (1-tx_i)^{-b_i} dt = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n). \quad (2.3)$$

Briefly, $F_D^{(n)}$ is a function of several (namely n) complex variables, defined by the power series

$$\begin{aligned} F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\ = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n} m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}, \end{aligned} \quad (2.4)$$

whenever $|x_1|, \dots, |x_n| < 1$ and by analytic continuation elsewhere. As a matter of notation, $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\dots(a+n-1)$ denotes the Pochhammer symbol (rising factorial).

2.2 $SU(N)$ without Hypermultiplets

Using the notation from the previous section, let us choose a canonical homology basis $B = \{\alpha_i, \beta^i\}_i$ for our curve, such that for any $\gamma \in B$ we have

$$\int_{\gamma} \lambda_{\text{SW}} = 2 \int_{e_i}^{e_j} \lambda_{\text{SW}}, \quad (2.5)$$

for some branch points e_i, e_j .

Geometrically speaking, the existence of such a basis means that if γ encircles e_i and e_j , no other e lies on the straight line connecting those two points, so that the original contour integral can be converted into twice the integral along that line.

However, for some configurations of the e 's no such basis might exist and therefore, unless stated otherwise, we shall explicitly *assume* its existence, treating it as an *hypothesis* for what follows.

As stated above, the quantity a_i (a_D^i) is obtained by integrating the Seiberg–Witten differential λ_{SW} along the cycle α_i (β^i) for which we indiscriminatingly write γ .

Thus we have

$$\begin{aligned}
a_i \quad \text{or} \quad a_D^i &= \int_{\gamma} \lambda_{\text{SW}} = 2 \int_{e_i}^{e_j} \lambda_{\text{SW}} \\
&= \frac{1}{\pi i} \int_{e_i}^{e_j} \prod_{k=0}^{N-1} (x - z_k) \prod_{\ell=1}^{2N} (x - e_{\ell})^{-\frac{1}{2}} dx \\
&= \frac{1}{\pi i} (e_j - e_i) \int_0^1 \prod_{k=0}^{N-1} (e_i - (e_i - e_j)t - z_k) \prod_{\ell=1}^{2N} (e_i - (e_i - e_j)t - e_{\ell})^{-\frac{1}{2}} dt \\
&= \frac{1}{\pi i} (e_j - e_i) \prod_{k=0}^{N-1} (e_i - z_k) (e_i - e_j)^{-\frac{1}{2}} (e_j - e_i)^{-\frac{1}{2}} \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^{2N} (e_i - e_j)^{-\frac{1}{2}} \times \\
&\quad \times \int_0^1 t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} \prod_{k=0}^{N-1} \left(1 - t \frac{e_i - e_j}{e_i - z_k} \right) \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^{2N} \left(1 - t \frac{e_i - e_j}{e_i - e_{\ell}} \right)^{-\frac{1}{2}} dt \\
&= (e_i - e_j)^{\frac{1}{2}} \prod_{k=0}^{N-1} (e_i - z_k) \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^{2N} (e_i - e_{\ell})^{-\frac{1}{2}} \times \\
&\quad \times F_D^{(3N-2)} \left(\underbrace{\frac{1}{2}, -1, \dots, -1}_{N \text{ parameters}}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{2N-2 \text{ parameters}}; 1; \underbrace{\left\{ \frac{e_i - e_j}{e_i - z_k} \right\}_k}_{N \text{ variables}}, \underbrace{\left\{ \frac{e_i - e_j}{e_i - e_{\ell}} \right\}_{\ell \neq i, j}}_{2N-2 \text{ variables}} \right), \tag{2.6}
\end{aligned}$$

where in the last line we have made use of the integral identity 2.3.

This is the main result of this section.

By a slight change in notation in this formula we can greatly shift emphasis towards a more ‘geometric’ viewpoint.

To this end we make the following definitions

$$\zeta := e_i - e_j, \tag{2.7}$$

$$\xi_{\nu} := \frac{e_i - e_j}{e_i - z_{\nu-1}}, \quad \nu \in \{1, \dots, N\}; \tag{2.8}$$

$$x_{\nu} := \frac{e_i - e_j}{e_i - e_{\nu}}, \quad \nu \in \{1, \dots, i-1, i+1, \dots, 2N\}. \tag{2.9}$$

Note that x_i is *not defined*.

Furthermore, let

$$b_{\xi} := \underbrace{(-1, \dots, -1)}_{N \text{ components}}, \tag{2.10}$$

$$b_x := \underbrace{\left(\frac{1}{2}, \dots, \frac{1}{2} \right)}_{2N-2 \text{ components}}. \tag{2.11}$$

Then the result 2.6 reads

$$a_i \quad \text{or} \quad a_D^i = \zeta \prod_{\nu=1}^N \xi_\nu^{-1} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{2N} x_\nu^{\frac{1}{2}} F_D^{(3N-2)}\left(\frac{1}{2}, b_\xi, b_x; 1; \{\xi_\nu\}_{\nu=1}^N, x_1, \dots, \widehat{x_j}, \dots, x_{2N}\right), \quad (2.12)$$

where $\widehat{}$ denotes omission. We remark that in total there are $3N - 2$ ‘ x ’-arguments to the F_D , x_i being undefined and x_j being omitted.

The promised shift in emphasis takes place if we now consider a_i and a_D^i as functions solely of ζ and the ξ ’s and x ’s, the domain of these ‘new’ functions being the subset of $\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{2N-2}$ which results if we regard ζ and the ξ ’s and x ’s as functions of the u ’s—the vacuum expectation values—and let the latter vary freely over the whole of \mathbb{C}^{N-1} .

Unfortunately it turns out that the described domain is rather complicated. Nevertheless, we believe that this viewpoint sheds some additional light on the structure of the moduli space.

The Lauricella function of type F_D has been used in the context of Seiberg–Witten theory before [12, 13]. However, the approach take there made use of conformal invariance of the Seiberg–Witten periods which leads to a different set of variables. The main problem with this older approach is that the variables involve crossing ratios of the branch points and zeroes of the Seiberg–Witten differential. This makes an analysis of singular points, as we will perform it in simple examples below, almost impossible.

2.3 $SU(N)$ with Hypermultiplets

As remarked in Section 2.1 the case where massive hypermultiplets are present brings about some modifications in the Seiberg–Witten differential and the curve on which it is defined. Namely, if we denote the gauge group by $SU(N_c)$ and if there are present N_f hypermultiplets with masses m_r , the curve-defining equation becomes

$$y^2 = A(x)^2 - B(x) := \left(x^{N_c} - \sum_{k=2}^{N_c} u_k x^{N_c-k} \right)^2 - \Lambda^{2N_c-N_f} \prod_{r=1}^{N_f} (x - m_r) = \prod_{i=1}^{2N_c} (x - e_i), \quad (2.13)$$

and the Seiberg–Witten differential then reads

$$\lambda_{\text{SW}} = \frac{1}{2\pi i} \frac{\prod_{\ell=0}^{N_c+N_f-1} (x - z_\ell)}{\prod_{i=1}^{2N} \sqrt{x - e_i} \prod_{j=1}^{N_f} (x - m_j)} dx, \quad (2.14)$$

where the z_ℓ ($\ell > 0$) now denote the zeros of $2A'(x)B(x) - A(x)B'(x)$, and again $z_0 = 0$.

Under the same conditions⁴ on our homology basis as in the previous section we obtain

⁴This is not entirely true, because now we must also make sure, that no m_r lies on the straight line connecting e_i and e_j and we also have to enlarge the homology basis to include cycles round the poles of the Seiberg–Witten differential as explained in the introduction. The corresponding scalar modes are suppressed here, since their calculation reduces to the evaluation of residues.

by a calculation absolutely analogous to that of 2.6

$$\begin{aligned}
a_i \quad \text{or} \quad a_D^i = & (e_i - e_j)^{\frac{1}{2}} \prod_{k=0}^{N_c+N_f-1} (e_i - z_k) \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N_c} (e_i - e_\ell)^{-\frac{1}{2}} \prod_{r=1}^{N_f} (e_i - m_r)^{-1} \times \\
& \times F_D^{(3N_c+2N_f-2)} \left(\frac{1}{2}, \underbrace{-1, \dots, -1}_{N_c+N_f \text{ parameters}}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{\substack{2N_c-2 \\ \text{parameters}}}, \underbrace{1, \dots, 1}_{N_f \text{ parameters}}; 1; \right. \\
& \left. \underbrace{\left\{ \frac{e_i - e_j}{e_i - z_k} \right\}_k}_{N_c+N_f \text{ variables}}, \underbrace{\left\{ \frac{e_i - e_j}{e_i - e_\ell} \right\}_{\ell \neq i,j}}_{2N_c-2 \text{ variables}}, \underbrace{\left\{ \frac{e_i - e_j}{e_i - m_r} \right\}_r}_{N_f \text{ variables}} \right).
\end{aligned} \tag{2.15}$$

This is the result for the case with massive hypermultiplets.

Of course one could also clean up the notation in this formula, but since we will not be making any use of it we shall pass this by.

3. Applications

In this section we discuss applications of the general formulas developed earlier. First, as a warm up, we show how some well known results (namely the asymptotics for the scalar modes $a(u)$ and $a_D(u)$ as $u \rightarrow 0$) for the $SU(2)$ case can be obtained quite easily. After that, we study the Argyres–Douglas \mathbb{Z}_3 -point in the $SU(3)$ case and address in detail the various issues raised in the introduction.

3.1 $SU(2)$

Let us calculate the scalar modes when the gauge group is $SU(2)$. Referring to the curve 2.1, here $N = 2$ and there is exactly one modulus u .

The z 's (cf. the line below eq. 2.2) are

$$z_0 = 0, \quad z_1 = 0 \tag{3.1}$$

and the branch points are

$$e_1 = \sqrt{u - \Lambda^2}, \tag{3.2}$$

$$e_2 = -\sqrt{u + \Lambda^2}, \tag{3.3}$$

$$e_3 = -\sqrt{u - \Lambda^2}, \tag{3.4}$$

$$e_4 = \sqrt{u + \Lambda^2}. \tag{3.5}$$

Taking this into account, we have the situation depicted in figure 1.

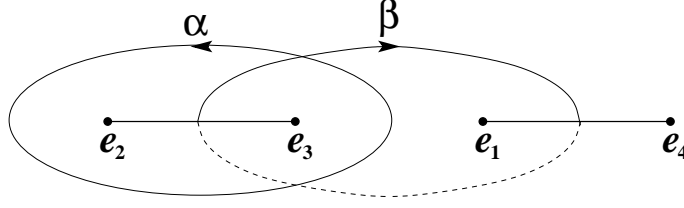


Figure 1: Homology basis for $SU(2)$

Therefore, using 2.6 (our homology basis $\{\alpha, \beta\}$ is shown in fig. 1),

$$\begin{aligned}
a(u) &= \int_{\alpha} \lambda_{\text{SW}} = -2 \int_{e_2}^{e_3} \lambda_{\text{SW}} \\
&= -\frac{e_2^2}{(e_2 - e_1)^{\frac{1}{2}}(e_2 - e_4)^{\frac{1}{2}}} F_D^{(4)}\left(\dots; \frac{e_2 - e_3}{e_2 - 0}, \frac{e_2 - e_3}{e_2 - 0}, \frac{e_2 - e_3}{e_2 - e_1}, \frac{e_2 - e_3}{e_2 - e_4}\right) \\
&\sim \frac{1}{\sqrt{8}} \sqrt{2u}, \quad u \rightarrow \infty,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
a_D(u) &= \int_{\beta} \lambda_{\text{SW}} = -2 \int_{e_3}^{e_1} \lambda_{\text{SW}} \\
&= -\frac{e_3^2}{(e_3 - e_2)^{\frac{1}{2}}(e_3 - e_4)^{\frac{1}{2}}} F_D^{(4)}\left(\dots; \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right) \\
&\sim -\frac{1}{\pi} \frac{e_3^2}{(e_1 - e_3)^{\frac{1}{2}}(e_3 - e_4)^{\frac{1}{2}}} \log\left(\frac{e_1 - e_3}{e_3 - e_2}\right), \quad u \rightarrow \infty \\
&\sim \frac{1}{\sqrt{8}} \frac{i}{\pi} \sqrt{2u} \log u, \quad u \rightarrow \infty,
\end{aligned} \tag{3.7}$$

where for $a_D(u)$ we have made use of equation A.11 in the appendix.

These results are of course well known.

The rigorous minded reader might want to consider the following paragraph, where we explain in a little more detail how we applied A.11 to obtain the asymptotic behavior of $a_D(u)$.

Details of previous derivation. For the following it is useful to write out in full the F_D in formula 3.7:

$$F_D^{(4)}\left(\frac{1}{2}, -1, -1, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right). \tag{3.8}$$

Since the first two arguments are identical and equal to 2 this reduces to

$$F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right), \tag{3.9}$$

as is easily seen from the power series representation 2.4. Using the power series even though $\frac{e_3 - e_1}{e_3 - 0} \equiv 2$ is legal, since the corresponding parameters are negative integers so

that this part of the series terminates after a finite number of terms, giving something polynomial in the argument.

Following up on the remark ‘something polynomial’ we obtain

$$\begin{aligned}
F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; \mathbf{2}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right) \\
= F_D^{(2)}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right) \\
- 2F_D^{(2)}\left(\frac{1}{2} + 1, \frac{1}{2}, \frac{1}{2}; 1 + 1; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right) \\
+ \frac{3}{8}\mathbf{2}^2 F_D^{(2)}\left(\frac{1}{2} + 2, \frac{1}{2}, \frac{1}{2}; 1 + 2; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\right).
\end{aligned} \tag{3.10}$$

Now, in the limit $u \rightarrow \infty$, $|\frac{e_3 - e_1}{e_3 - e_2}| \rightarrow \infty$ and $|\frac{e_3 - e_1}{e_3 - e_4}| \nearrow 1$, the divergence being faster than the convergence. Therefore we can employ the analytic continuation formula A.11. Doing so, one easily verifies that the first asymptotic in 3.7 is correct. The expression in the fourth line is asymptotically equal to the one before and since the relation \sim is transitive everything is in perfect order.

3.2 Argyres–Douglas’ \mathbb{Z}_3 -point

In this section we apply our main result 2.6 to the \mathbb{Z}_3 -point discovered by Argyres and Douglas (cf. [2, 3]). Also, [14, 31, 29], and references cited therein, have made contributions to this subject. The case with hypermultiplets was considered in [11, 30].

Roughly speaking, the Argyres–Douglas point is interesting because it provides us with an example of a theory in which the BPS spectrum contains a pair of dual particles, i.e. particles where one is electrically and the other magnetically charged, which becomes simultaneously massless. Furthermore, at such points in moduli space, the theory becomes superconformal which, of course, has important consequences [23].

3.2.1 Vanishing of Scalar Modes

The \mathbb{Z}_3 -point is the e -configuration in the $SU(3)$ case determined by the choice of vev’s $u = 0$, $v = \Lambda^3$ (for completeness, we mention that there exist other choices which also lead to ‘ \mathbb{Z}_3 -points’). Our aim is to apply our formula 2.6 to this case.

Therefore, following Argyres and Douglas, we write

$$u = \delta u, \quad v = \Lambda^3 + \delta v. \tag{3.11}$$

We shall restrict ourselves to (real) $\delta u < 0$ and (real) $\delta v > 0$. We remark on these hypotheses in Section 3.2.2 below.

Referring to the curve 2.1, the e ’s (branch points) are the zeros of

$$(x^3 - \delta u x - (\Lambda^3 + \delta v))^2 - \Lambda^6. \tag{3.12}$$

They are easily seen to be separated into two classes: Those which are the zeros of $p_1(x) = x^3 - \delta u x - \delta v$ and those which are the zeros of $p_2(x) = x^3 - \delta u x - \delta v - 2\Lambda^3$.

Put $P = \text{sgn}(-\delta v/2) \sqrt{|-\delta u/3|}$ and $\beta = \frac{1}{3} \text{arsinh} \frac{-\delta v/2}{P^3}$.

Then, under our assumptions on the δ 's, the zeros of p_1 are (see e.g. [6])

$$\begin{aligned} e_1 &= -2P \sinh \beta, \\ e_2 &= P(\sinh \beta - i\sqrt{3} \cosh \beta), \\ e_3 &= P(\sinh \beta + i\sqrt{3} \cosh \beta). \end{aligned} \tag{3.13}$$

We shall not need explicit formulas for the zeros of p_2 . It will suffice to know that whenever the δ 's are small, the zeros of p_2 (call them e_4, e_5, e_6) will be near the third roots of $2\Lambda^3$.

The resulting configuration, along with our choice of homology basis, is visualized in Figure 2. Before writing down our expressions for the a 's and a_D 's we also need to know

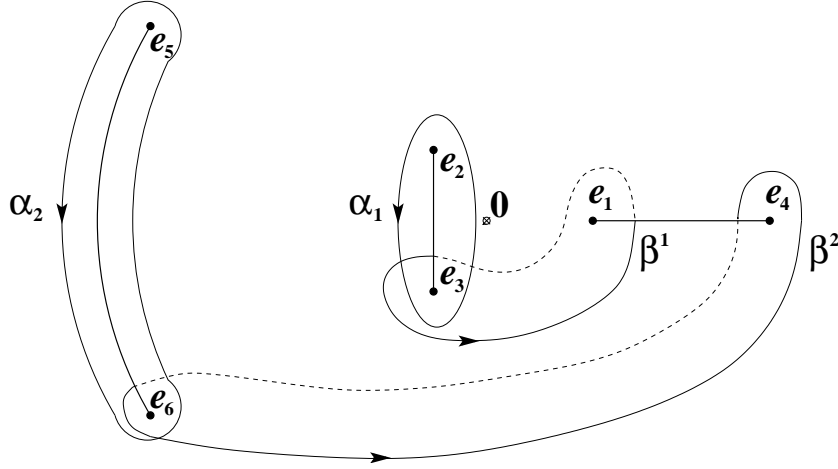


Figure 2: Homology basis for $SU(3)$

that the z 's (recall the definition just below eq. 2.2) now are

$$z_0 = 0, \quad z_1 = \sqrt{\delta u/3}, \quad z_2 = -\sqrt{\delta u/3}. \tag{3.14}$$

In the study of the \mathbb{Z}_3 -point one is mostly interested in a_1 and a_D^1 because it are these two dual quantities which simultaneously vanish at the \mathbb{Z}_3 -point. This, of course, is related to the fact that as the δ 's tend to 0 the cycles α_1 and β^1 contract to points because e_1, e_2, e_3 tend to 0.

As functions of δu and δv , i. e. near the Argyres–Douglas point, the scalar modes a_1, a_D^1 are given by (cf. eq. 2.6)

$$\begin{aligned} a_1 &= 2 \int_{e_2}^{e_3} \lambda_{\text{SW}} = \frac{e_2(e_2^2 - \delta u/3)}{(e_2 - e_1)^{\frac{1}{2}}(e_2 - e_4)^{\frac{1}{2}}(e_2 - e_5)^{\frac{1}{2}}(e_2 - e_6)^{\frac{1}{2}}} \times \\ &\times F_D^{(7)}\left(\dots; \frac{e_2 - e_3}{e_2 - 0}, \frac{e_2 - e_3}{e_2 - \sqrt{\delta u/3}}, \frac{e_2 - e_3}{e_2 + \sqrt{\delta u/3}}, \frac{e_2 - e_3}{e_2 - e_1}, \frac{e_2 - e_3}{e_2 - e_4}, \frac{e_2 - e_3}{e_2 - e_5}, \frac{e_2 - e_3}{e_2 - e_6}\right), \end{aligned} \tag{3.15}$$

and

$$a_D^1 = 2 \int_{e_3}^{e_1} \lambda_{\text{SW}} = \frac{e_3(e_3^2 - \delta u/3)}{(e_3 - e_2)^{\frac{1}{2}}(e_3 - e_4)^{\frac{1}{2}}(e_3 - e_5)^{\frac{1}{2}}(e_3 - e_6)^{\frac{1}{2}}} \times \\ \times F_D^{(7)}\left(\dots; \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - \sqrt{\delta u/3}}, \frac{e_3 - e_1}{e_3 + \sqrt{\delta u/3}}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}, \frac{e_3 - e_1}{e_3 - e_5}, \frac{e_3 - e_1}{e_3 - e_6}\right). \quad (3.16)$$

The next thing we shall do is to examine the behavior of a_1 and a_D^1 as the δ 's both tend to 0. There are several meanings one can attach to the phrase ‘tend to 0.’ One possible meaning is the usual one from the calculus of several (complex) variables, where both variables are considered as *independent* of each other. Another possible meaning, and this is often done in physics, is to consider both variables as *dependent*, say $\delta v = -\delta u$ (recall our assumptions about the δ 's). We will follow the latter course (cf. our general remarks in Section 3.2.2 below). This ensures that both variables will be of the same order of smallness during the limiting process.

We first consider a_1 : As $\delta u \rightarrow 0$, $\delta u < 0$, the arguments to the F_D in eq. 3.15 behave as follows.

$$\begin{aligned} \frac{e_2 - e_3}{e_2 - 0} &\rightarrow \frac{1}{2}(3 - i\sqrt{3}), \\ \frac{e_2 - e_3}{e_2 - \sqrt{\delta u/3}} &\rightarrow \frac{1}{2}(3 - i\sqrt{3}), \\ \frac{e_2 - e_3}{e_2 + \sqrt{\delta u/3}} &\rightarrow \frac{1}{2}(3 - i\sqrt{3}), \\ \frac{e_2 - e_3}{e_2 - e_1} &\rightarrow \frac{1}{2}(1 - i\sqrt{3}), \\ \frac{e_2 - e_3}{e_2 - e_4} &\rightarrow 0, \\ \frac{e_2 - e_3}{e_2 - e_5} &\rightarrow 0, \\ \frac{e_2 - e_3}{e_2 - e_6} &\rightarrow 0. \end{aligned} \quad (3.17)$$

Furthermore, *the prefactor of the F_D in eq. 3.15 tends to 0*. These limits were obtained using MATHEMATICA. We will now prove that the value of the F_D at the limits just written down is a complex number (i. e., that it is defined for that particular constellation of arguments).

First observe that

$$F_D^{(7)}(\dots, 0, 0, 0) = F_D^{(4)}(\dots), \quad (3.18)$$

and

$$\begin{aligned} F_D^{(4)}\left(\frac{1}{2}, -1, -1, -1, \frac{1}{2}; 1; \frac{1}{2}(3 - i\sqrt{3}), \frac{1}{2}(3 - i\sqrt{3}), \frac{1}{2}(3 - i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})\right) \\ = F_D^{(2)}\left(\frac{1}{2}, -3, \frac{1}{2}; 1; \frac{1}{2}(3 - i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})\right). \end{aligned} \quad (3.19)$$

Since -3 is a negative integer this last F_D reduces to a polynomial in $\frac{1}{2}(3 - i\sqrt{3})$ (see Appendix A), the coefficients being rational numbers multiplied by expressions of the form ${}_2F_1\left(\frac{1}{2} + n, \frac{1}{2}; 1 + n; \frac{1}{2}(1 - i\sqrt{3})\right)$, where n is some nonnegative integer (${}_2F_1$ is the Gaussian hypergeometric function). Since for our constellation of parameters the only singularity of ${}_2F_1$ on the unit circle is at 1, these ‘expressions’ reduce to (finite) complex numbers. Thus, as promised, the F_D in eq. 3.15 converges at the limits of its arguments.

Therefore (recall that the prefactor tends to 0), a_1 tends to 0, as $\delta u \rightarrow 0$, $\delta u < 0$.

We now consider a_D^1 : Referring to eq. 3.16 one finds that the arguments of the F_D in that equation, taken in the same order as those of the F_D in eq. 3.15, tend to the same limits as indicated in eq. 3.17. Following exactly the same arguments as for a_1 we find that the F_D in the expression for a_D^1 tends to the same (finite) complex number as the one in eq. 3.15. Also, the prefactor in eq. 3.16 tends to 0 as $\delta u \rightarrow 0$, $\delta u < 0$.

Therefore, a_D^1 tends to 0 as $\delta u \rightarrow 0$, $\delta u < 0$.

This vanishing of a_1 and a_D^1 reproduces the result stated by Argyres and Douglas [2] in the special limit (recall our restrictions on the δ ’s) we have considered.

3.2.2 General Discussion

The reader might ask why in the previous section we imposed so peculiar restrictions on δu and δv . In the present section, we wish to address precisely this question.

Partly the answer is simple: We imposed these conditions so that we were able to consider the limit of the scalar modes at the Argyres–Douglas point in a straight-forward manner. For instance, we know for sure that it is possible to consider the case $\delta u > 0$, $\delta u = \delta v$, $\delta u \rightarrow 0$,⁵ and the analysis of this is exactly the same as in the case we have presented in detail (even the arguments of the F_D ’s tend to the same limits). However, there is a complication involved. Now one has to look at the discriminant of p_1 ; if it is 0, a degeneration of the ‘inner’ e ’s (cf. Fig. 2) takes place ($e_2 = e_3$, moreover, $e_2 = -\sqrt{-\delta u/3}$), and if it is negative, the homology basis of Fig. 2 is inadequate for our formulas because then all ‘inner’ e ’s are real. One does away with the trouble of the discriminant in this case by setting $\delta u = \delta v$ (this implies positivity of the discriminant). The moral is that it is possible to impose different restrictions on the δ ’s than we have done in this paper.

In the general case of arbitrary complex $\delta u, \delta v$ with an arbitrary approach to 0 (as in standard calculus) the complications pile up: Firstly, there is the technical problem of keeping track of third roots of complex numbers. This is very difficult, since the expressions involved in the usual Cardano’s formula are intrinsically discontinuous due to branch cuts; also, multivaluedness becomes a burden. Secondly, there is the problem that one cannot choose a homology basis once and for all. For instance, if $\delta v > 0$, $\delta u < 0$ the configuration of the ‘inner’ e ’s looks like that of those in Fig. 2, only reflected through the origin. Of course this necessitates the use of a different homology basis than the one shown in Figure 2. If the δ ’s can arbitrarily approach 0, then it is clear that one could start with a homology basis like the one in Fig. 2 and end up with the need to choose a different one. Our formulas are not suited for such a case.

⁵Again, a_1 and a_D^1 vanish.

The question arises: Do a_1, a_D^1 vanish at the Argyres–Douglas point regardless of exactly how the δ ’s tend to 0? If they should happen to be continuous functions the answer is affirmative: Yes! However, we do not deem their continuity as self-evident, since the e ’s depend on the δ ’s in a rather complicated fashion. On some Riemann-surface a_1, a_D^1 most likely *are* continuous. But that seems to be something different.

It is tautological to say that if $a_1(\delta u, \delta v), a_D^1(\delta u, \delta v)$ should happen to be discontinuous at $(0, 0)$, then there would exist some approach of the δ ’s to 0 which would not yield a vanishing limit.

In light of the particular properties of the Lauricella functions under analytic continuation, we point out the following: The periods a_1 and a_D^1 are given in terms of Lauricella functions $F_D^{(7)}$ of seven arguments. As can be inferred from eqs. 3.15 and 3.16, the arguments depend on the branch points. In our conventions, the three branch points e_1, e_2 and e_3 are all located in a small neighbourhood around 0, if we approach the Argyres–Douglas point. However, generically, they are farther away from 0 than the two other zeroes of the Seiberg–Witten differential $z_{1,2} = \pm\sqrt{u/3}$. It is an easy task to convince oneself, that appropriate choices of δu and δv can invert this situation, such that some of the ‘small’ branch points are closer to zero than the z ’s. Of course, then the situation is possible where a branch point coincides with one of the z ’s before the point 0 is approached. Also, it can be arranged that two of the three ‘small’ branch points coincide first, before the three finally approach zero. All these situations fall in the class of singular sub-manifolds.

Let us look specifically at the fourth argument of the periods. In the notation introduced in section 2.2 it reads $x_1 = \frac{e_2 - e_3}{e_2 - e_1}$ for $a_1(\delta u, \delta v)$ or $x_2 = -\frac{e_3 - e_1}{e_2 - e_3}$ for $a_D^1(\delta u, \delta v)$, respectively. Clearly, depending on how the three ‘small’ branch points approach each other and 0, the moduli of the two arguments are related as $|x_1| = 1/|x_2|$ and hence are located around one of the following pairs of singular points: $(1, 1)$, $(0, \infty)$, or $(\infty, 0)$. Since this fourth argument is associated with a parameter b_x , which is not a negative integer, the corresponding Lauricella function is not simply polynomial in this argument. Thus, in all four cases will we need at least one analytic continuation, either of a_1 or of a_D^1 . Due to the specific values of the parameters, each of these analytic continuations will involve a logarithmic divergence of the involved Lauricella function, killed off only by the prefactor.

Next, we look at the last three arguments. Generically, all three approach 0 from three different directions. However, in particular if Λ is allowed to be small and complex, the ‘large’ branch points might run into each other first, e.g. on the sub-manifold characterized by $\{\delta u, \delta v, \Lambda, w : \delta u = 3w^2, \delta v + 2\Lambda^3 = -2w^3\}$. This is then exactly the difficult situation that the singular point $(x_{1,2}, 0, 0, 0)$ is a common point of various sub-manifolds $x_k = x_l$ for $k, l \in \{4, 5, 6\}$. Similarly, the first three arguments close in on one common singular point, but this is not really a difficulty, since the corresponding parameters render the Lauricella functions polynomial in their first three variables. Hence, analytic continuation in one of the first three variables is only necessary in the true limit of the Argyres–Douglas point itself, where these variables all tend to infinity. For this, one should note that the zeroes of λ_{SW} are generically closer to 0, corresponding to the Argyres–Douglas point, than the ‘small’ branch points.

Therefore, depending on the choice of the δ 's, we may move on or across singular sub-manifolds which will make it necessary to perform the appropriate analytic continuations. In particular, if we have to analytically continue in more than one variable, say x_5 and x_6 , great care has to be taken that for an analytic continuation around, say, the point $(1, 1)$ we need *two* different continuations, namely for the patches $|x_5| < |x_6|$ and $|x_5| > |x_6|$, respectively. These can be found along the lines set out in the appendix, but they are more complicated than the examples provided there. The discriminant of the curve y^2 tells us where branch points coincide. A special locus of that kind is $\{\delta u, \delta v : 27(\delta v)^2 = 4(\delta u)^3\}$, where two of the three 'small' branch points become equal, say e_2 and e_3 . Then $a_1(\delta u, \delta v)$ is well defined around the singular point $(0, 0, 0, 0)$ for its last four variables, while $a_D^1(\delta u, \delta v)$ has to be analytically continued around $(\infty, 0, 0, 0)$ according to A.8. The situation gets more involved, if we allow Λ to vary arbitrarily or if we include massive hypermultiplets. Then, almost any singular sub-manifold can be reached, and the distinction between 'small' and 'large' branch points may cease to exist, leading to higher-dimensional singular sub-manifolds.

Lastly, we would like to emphasize that eqs. 3.15 and 3.16 are, of course, valid without any particular restrictions on the δ 's other than that all relevant expression be defined.

4. Conclusions and Outlook

We have derived general formulas for the scalar modes in Seiberg–Witten theory for gauge group $SU(N)$ in the absence as well as in the presence of massive hypermultiplets. These formulas involve the Lauricella function $F_D^{(n)}$, a multiple hypergeometric function. The advantage of our approach is threefold. Firstly, our formulas can be worked out directly, without the need to first find the correct system of Picard–Fuchs differential equations and its solutions. Secondly, our formulas work equally well for the case with (massive) hypermultiplets, putting them on equal footing with the ordinary case. Finally, one can immediately extract a well-defined power series expansion for any BPS period from our results, as long as the corresponding homology cycle is the smallest one.⁶ Thus, in any given patch of moduli space, the lightest BPS state for this patch can be obtained in a straight-forward manner.

Subsequently, we have applied the formula for the case without hypermultiplets to the Argyres–Douglas point for gauge group $SU(3)$. Specifically, we have considered the limits of the scalar modes corresponding to the cycles α_1, β^1 as the vacuum expectation values u, v tend—in a special way—to those values which define the Argyres–Douglas point. In accord with the result of Argyres and Douglas we have found that the limit of both these modes is indeed zero.

As already pointed out, this raises the question of what happens when u, v *arbitrarily* tend to the prescribed values. We cannot tell for sure. To begin with, we think it is conceivable that there exists some mode of approach which yields a limit other than 0. This possibility is intimately connected with—indeed, is equivalent to—the question whether the

⁶This is a sufficient condition. Of course, easily computable power series expansions exist for many more cases.

scalar modes are continuous functions of the vacuum expectation values. With the existence of phase transitions in Seiberg–Witten theory in mind, we believe that it is necessary to prove the continuity of the scalar modes rather than just assume it.

Even if the scalar modes should turn out to be continuous it is quite puzzling to observe that in the special case we have considered the Lauricella functions involved in our formulas tend to the same values, so that the vanishing of the scalar modes is determined by the vanishing of the prefactors. What is puzzling about this is that it looks as if a_1, a_D^1 become linearly dependent at the Argyres–Douglas point. In light of the theory of Lauricella functions we conjecture that the set of BPS periods alone is not exhaustive to define the scalar modes everywhere in moduli space. Furthermore, if indeed a_1 and a_D^1 become linearly dependent, then we might happen to be sitting on the curve of marginal stability, meaning that not both of the corresponding BPS modes actually exist.

As discussed, it is a known problem in the theory of generalized hypergeometric functions of several variables that analytic continuation around singular points becomes extremely difficult. The reason is the appearance of singular sub-manifolds in case more than one of the variables approaches a singular point, or more than two variables approach each other. It is interesting to note the following: In case of gauge group $SU(N)$ without the presence of hypermultiplets, the hyperelliptic curve can be written as $y^2 = A(x) - \Lambda^{2N}$. The Seiberg–Witten differential then is proportional to $\frac{1}{y}A'(x)xdx$. Thus, the zeroes of the Seiberg–Witten differential are given precisely by those sub-manifolds, where $y^2 = A(x)^2 - \Lambda^{2N}$ has a double zero, i.e. where we run into difficulties with ordinary analytic continuation techniques.⁷ The special zero $x = 0$ naturally arises in the semi-classical regime $\Lambda \rightarrow 0$. The reader should keep in mind that the zero of λ_{SW} which we studied in our treatment of the Argyres–Douglas point is actually the zero $x = 0$ which always (not only in the semi-classical regime) arises for $u_N = \Lambda^N$ in $A(x) = x^N - \sum_{k=2}^N u_k x^{N-k}$, for which $y^2 = 0$. Thus, this is actually such a double zero of y^2 .

In our approach, zeroes of the Seiberg–Witten differential appear on equal footing with the branch points of y and the poles, in case massive hypermultiplets are included. However, when explicit computations of the BPS periods are needed, the zeroes can be used to decompose the Lauricella functions of n arguments into linear combinations of such with fewer arguments, since the dependency on a zero $x = z_\ell$ is always only polynomial. On the other hand, it has been pointed out [18, 24, 5] that the zeroes of the Seiberg–Witten differential have a physical meaning. Our analysis suggests that the zeroes of the Seiberg–Witten differential are precisely related to the sub-manifolds which delimit the hypercones in which analytic continuations can be uniformly defined. Schulze and Warner [24] introduce geodesic integration paths which start or end in zeroes of the Seiberg–Witten differential. In principle, we can extend our analysis by *formally* enlarging our homology to include the zeroes of the Seiberg–Witten differential as further branch points of a thus implicitly defined Riemann surface. This is usually achieved by deforming the associated parameters of the Lauricella functions slightly away from negative integers and taking appropriate limits afterwards. However, there is presumably an alternative point

⁷The situation is more complicated in the presence of massive hypermultiplets.

of view. As we have shown, analytic continuations of the Lauricella functions of type D cannot be obtained everywhere within the set of these functions alone. Enlarging the set of functions confronts one with new types of generalized hypergeometric functions which do not possess integral representations in terms of line integrals or Pochhammer double loop integrals, where the loop encloses two and only two singular points. It is conjectured in the literature (see e.g. [10]) that one needs at least the set of all double circuits and all three-foil loops, which are selfintersecting loops which enclose two or three disjoint subsets of singular points, respectively.

We have used precisely the form of the Seiberg–Witten differential which arises in the context of string theory as a local form of a metric on the auxiliary Riemann surface Σ . Although we have followed the traditional approach where BPS states are defined in terms of periods of homology elements, we have found that the set of homology cycles is neither sufficient, nor everywhere suitable, to describe the BPS spectrum. We do not know how to interpret analytic continuations of the scalar modes, which do not have nice and simple integral representations, but it strikes us as remarkable that regions with given analytic continuations have the form of hypercones delimited by sub-manifolds, which are parametrized by the zeroes of the Seiberg–Witten differential. The theory of Lauricella functions seems at present not well developed enough to fully understand how geodesic integrals between zeroes of the Seiberg–Witten differential can be expressed in our approach, i.e. in terms of Lauricella functions and in particular their analytic continuations valid on the singular sub-manifolds. It is worth noting in this context that Lauricella functions do have the special property that line integrals of the Seiberg–Witten differential to an arbitrary point in the complex plane can easily be written down, namely

$$\begin{aligned} \int_0^z du u^{a-1} (1-u)^{c-a-1} \prod_{i=1}^n (1-ux_n)^{-b_n} &=: \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_z\text{inc}F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\ &= \frac{z^a}{a} F_D^{(n+1)}(a, b_1, \dots, b_n, 1+a-c; a+1; x_1, \dots, x_n, z), \end{aligned} \quad (4.1)$$

which is known also as the incomplete Lauricella function. If $|z| > 1$, this expression has to be replaced by the appropriate analytic continuation.

Therefore, linking our analysis to the study of geodesic horizons would be a most interesting direction for future research. More work is needed in order to fully understand the analytic properties of the Lauricella system $F_D^{(n)}$. Furthermore, we have a price to pay for being able to compute periods of the Seiberg–Witten form explicitly: We have to change the variables from the vev’s of the scalar field to quotients of differences of branch points. This is a highly non-linear and multi-valued map. However, we would like to point out that the alternative, where one works with the vev’s u_k , means to find the system of Picard–Fuchs equations and to solve it. This has been achieved in the case of gauge group $SU(3)$, [16, 17], but the results are given in terms of the Appell function F_4 of two arguments (which are relatively simple functions of u_1 and u_2), which does not admit a nice integral representation. Furthermore, the problem of finding analytic continuations is similarly complicated. There are regions in moduli space where simple monodromy transformations will yield the desired results, but a complete set of analytic continuations

cannot be obtained in this way. Of course, the reason is again the existence of higher-dimensional singular sub-manifolds, here in \mathbb{C}^2 spanned by u_1 and u_2 . In addition, the Appell function F_4 and its generalization to more than two variables has an even worse structure under analytic continuation than the Lauricella $F_D^{(n)}$ system. Therefore, we believe that our approach offers several advantages when we wish to investigate Seiberg–Witten theory in all of its moduli space.

First of all, the Lauricella system $F_D^{(n)}$ together with the related functions $D_{p,q}^{(n)}$ are the simplest of all the generalized hypergeometric systems of several variables. It might be tedious, but it is possible to compute analytic continuations for them such that one can access all of moduli space, not only the semi-classical regime. Moreover, the formulas are explicit and thus allow direct computations of the scalar modes.

Secondly, the geometrical setup behind our approach makes the appearance of singular sub-manifolds very transparent. Since we realize the Seiberg–Witten curve as a ramified covering of the complex plane, the branch points, zeroes and poles of the Seiberg–Witten form have a direct meaning. However, we lose the direct relation to the vev’s of the scalar field. It depends on what one wishes to consider as ‘fundamental.’ From the point of view of effective field theory, the vev’s are the physically given parameters, and the Riemann surface is an auxiliary construction. From the vista of string theory, the Riemann surface has a certain reality. The Seiberg–Witten form is then uniquely fixed in the local form we used and which serves as a metric on Σ .

What remains open is the question how BPS states should really be characterized. The traditional approach, which also was our starting point, is via homology cycles of the Riemann surface Σ . Vanishing cycles then indicate BPS states which become massless. Other approaches emphasize the role of geodesic integration paths and zeroes of the Seiberg–Witten form. We believe that our approach is well suited to both points of view. It even suggests that just which characterization is more appropriate depends on the region of moduli space under consideration. A more detailed analysis of this question is very important and will be investigated in a forthcoming work.

Acknowledgments

N.A. thanks Christian Sämann for useful discussions. M.F. thanks Albrecht Klemm and Werner Nahm for useful discussions. The work of M.F. is supported by the European Union network HPRN-CT-2002-00325 (EUCLID) and in part by the string theory network (SPP no. 1096), Fl 259/2-2, of the Deutsche Forschungsgemeinschaft.

A. Lauricella $F_D^{(n)}$

The purpose of this appendix is to collect various information pertaining to Lauricella $F_D^{(n)}$. Some of the formulas have already made their appearance in one or the other section of the main text. The major reference for Lauricella $F_D^{(n)}$ and other ‘multiple hypergeometric functions’ is [10], from which we will cite freely.

A.1 The Definition

Lauricella $F_D^{(n)}$ is a function of n complex variables and $n + 2$ parameters, defined by the power series

$$F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n} m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}, \quad (\text{A.1})$$

whenever $|x_1|, \dots, |x_n| < 1$ and by analytic continuation elsewhere. The symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the so-called Pochhammer symbol.

The function $F_D^{(n)}$ has the integral representation

$$\int_0^1 t^{a-1} (1-t)^{c-a-1} \prod_{i=1}^n (1-tx_i)^{-b_i} dt = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n), \quad (\text{A.2})$$

if $\text{Re}(a)$ and $\text{Re}(c-a)$ are positive. This is proved using the binomial theorem. See [13], Appendix B for a detailed derivation.

A number of facts can be read off the power series representation A.1:

1. If one of the variables of $F_D^{(n)}$, say x_i , is equal to 0, then the $F_D^{(n)}$ reduces to a $F_D^{(n-1)}$:

$$\begin{aligned} F_D^{(n)}(a, b_1, \dots, b_i, \dots, b_n; c; x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \\ = F_D^{(n-1)}(a, b_1, \dots, \widehat{b_i}, \dots, b_n; c; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \end{aligned} \quad (\text{A.3})$$

where $\widehat{}$ denotes omission.

2. If two variables have equal values, say $x_i = x_j$, a similar reduction takes place:

$$\begin{aligned} F_D^{(n)}(a, b_1, \dots, b_i, \dots, b_j, \dots, b_n; c; x_1, \dots, x_i, \dots, x_j, \dots, x_n) \\ = F_D^{(n-1)}(a, b_1, \dots, b_{i-1}, b_i + b_j, b_{i+1}, \dots, \widehat{b_j}, \dots, b_n; c; x_1, \dots, x_i, \dots, \widehat{x_j}, \dots, x_n). \end{aligned} \quad (\text{A.4})$$

3. If b_i is a negative integer, then the part of the series corresponding to x_i terminates after a finite number of terms (because $(b_i)_n = 0$ for $n > |b_i|$) and thus reduces to a polynomial in x_i . In this case, the modulus of x_i is immaterial for the validity of A.1.

If $|x_i| \geq 1$, for some $i \in \{1, \dots, n\}$, then the power series representation A.1 is not valid anymore. Rather, one must have recourse to an analytic continuation of $F_D^{(n)}$. This is effected by writing

$$\begin{aligned} F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\ = \sum_{m_1=0}^{\infty} \dots \sum_{m_{i-1}=0}^{\infty} \sum_{m_{i+1}=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n} \prod_{\ell \neq i} (b_\ell)_{m_\ell}}{(c)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n} \prod_{\ell \neq i} m_\ell!} \prod_{\ell \neq i} x_\ell^{m_\ell} \times \\ \times {}_2F_1\left(a + \sum_{\ell \neq i} m_\ell, b_i; c + \sum_{\ell \neq i} m_\ell; x_i\right), \end{aligned} \quad (\text{A.5})$$

and employing a suitable continuation formula for the Gaussian hypergeometric function ${}_2F_1$. Erdélyi et al. [9] as well as Becken and Schmelcher [4] give several continuation formulas. Especially the latter reference appears to be quite exhaustive. Also, see the next section A.2.

Nevertheless, sometimes one can get along without any explicit continuation formula, just as we have done in Section 3.2.

A.2 Analytic continuation of Lauricella functions

The Seiberg–Witten periods are analytic functions ‘everywhere’ in the moduli space, i.e. for generic values of either the vacuum expectation values u_k or the branch points e_ℓ . However, it is clear that this is not necessarily the case in the singular regions where one or more branch points e_ℓ become identical. In fact, a typical feature of a dual pair (a, a_D) of Seiberg–Witten periods corresponding to a dual pair of homology cycles (α, β) is that with a vanishing cycle $\alpha \rightarrow 0$, only $a(u)$ becomes small, while the dual period $a_D(u)$ diverges logarithmically.

Precisely this is it what makes the Argyres–Douglas point so interesting: a pair of dual periods *simultaneously* become small, i.e. two particles dual to each other under some sort of electromagnetic duality, simultaneously become massless. As a consequence of this the theory is conformally invariant in this regime.

There is another interesting fact about such points in moduli space, where intersecting homology cycles vanish at the same time. If one uses the branch points as the natural coordinates to parametrize the theory, it is known from the theory of generalized hypergeometric functions that a complete set of analytic continuations cannot be given entirely in terms of the same class of functions one starts out with (cf. [10]). This is to be contrasted with the well-known result that the ordinary Gaussian hypergeometric function admits analytic continuations everywhere in the complex plane, which again can be expressed as linear combinations of Gaussian hypergeometric functions (cf., e.g. [4]). This can be used, for example, to find analytic continuation formulas for the Lauricella $F_D^{(n)}$ function, as long as we only need to continue one of its arguments outside the unit circle of convergence. As soon as we wish to have more than one argument outside the unit circle, things become complicated. In case of the Lauricella $F_D^{(n)}$ function, one is confronted with the following problem:

Within its region of convergence, the Lauricella $F_D^{(n)}$ series can be represented in the form of an Euler–type integral, i.e. an integral along a simple loop, which we choose to be one of the homology cycles. More generally, Pochhammer double loops might be admitted as well. The point is, that only two of the singular points of the differential are enclosed by the loop. The thus defined functions possess analytic continuations which, for generic values of the parameters, can again be given in terms of multi-variable power series (perhaps multiplied by a common fractional power). However, such analytic continuations are only valid outside the unit ball within cones delimited by the singular hyperplanes, given by coinciding singular points. Thus, one needs a considerably larger set of analytic continuations than in the single-variable case. Moreover, not all of these analytic continuations can be represented by Euler–type integrals. This raises the physically relevant question what

the meaning of the Seiberg–Witten periods then is. As long as they can be understood as contour integrals along homology basis elements, they represent the mass of particles with a charge fixed by the corresponding homology element. But what would the meaning be, if no such simple contour existed? Extton [10] mentions that at least so-called three-foil loops are necessary to be able to represent a full set of analytic continuations in terms of integrals. Three-foil loops are three times self-intersecting loops which enclose three different sets of singular points.

The linearity of the integral guarantees that, in principle, we could eliminate the internal dependence on the zeros of the Seiberg–Witten differential by decomposing the integrand. The resulting Lauricella functions of fewer arguments depend solely on the branch points via the quotients of differences of them, as introduced earlier. However, generic small variations of a subset of branch points which are in a small neighborhood around a physically interesting singular point such as the Argyres–Douglas point, have to be treated carefully. On the one hand, we might need several analytic continuations, because the small variation crosses a boundary of the cone of convergence, which coincides with the situation that the initial choice of a homology basis ceases to be valid. On the other hand, we even might run into a region where no Euler–type integral representation exists.

In order to study the analytic continuations of the Lauricella functions of type F_D , one needs a further class of related functions, defined by the expansions

$$D_{p,q}^{(n)}(a, b_1, \dots, b_n; c, c'; x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_{p+1}+\dots+m_n-m_1-\dots-m_p} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_{q+1}+\dots+m_n-m_1-\dots-m_p} c'_{m_{p+1}+\dots+m_q} m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}, \quad (\text{A.6})$$

where $0 \leq p \leq q \leq n$. It is important to note that these functions, which appear in the analytic continuations of the Lauricella $F_D^{(n)}$ functions, do not possess Euler–type integral representations. The simplest known integral representation is in fact a Pochhammer double loop integral involving a Lauricella function in its kernel, namely

$$\begin{aligned} & \frac{(2\pi i)^2}{\Gamma(a)\Gamma(a')\Gamma(2-a-a')} D_{p,q}^{(n)}(a+a'-1, b_1, \dots, b_n; c, c'; x_1, \dots, x_n) \\ &= \int du (-u)^{-a'} (u-1)^{-a} F_D^{q-p}(a', b_{p+1}, \dots, b_q; c'; \frac{x_{p+1}}{u}, \dots, \frac{x_q}{u}) \times \\ & \times D_{n-q+p, n-q+p}^{(p)}(a, b_{q+1}, \dots, b_n, b_1, \dots, b_p; c, c'; \frac{x_{q+1}}{1-u}, \dots, \frac{x_n}{1-u}, \frac{x_1}{1-u}, \dots, \frac{x_p}{1-u}), \quad (\text{A.7}) \end{aligned}$$

where the Pochhammer double loop encircles 0 and 1. We will now give three cases of analytic continuations. Other cases can be obtained in a similar way. One starts with the still simple case that only one argument is either close to 1 or ∞ . This case can be developed along the lines set out in [10] by rewriting the multiple series in such a way that the innermost summations is replaced by ordinary Gaussian hypergeometric series, for which the analytic continuation is known, see eq. A.5 above. This yields the following

results: For the region near infinity, i.e. $1/|x_n| < 1$, the analytic continuation reads

$$\begin{aligned}
& F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\
&= \frac{\Gamma(c)\Gamma(b_n - a)}{\Gamma(b_n)\Gamma(c - a)} (-x_n)^{-a} F_D^{(n)}(a, b_1, \dots, b_{n-1}, 1 - c + a; 1 - b_n + a; \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n}) \\
&+ \frac{\Gamma(c)\Gamma(a - b_n)}{\Gamma(a)\Gamma(c - b_n)} (-x_n)^{-b_n} D_{1,1}^{(n)}(a - b_n, b_n, b_1, \dots, b_{n-1}; c - b_n, c - b_n; \frac{1}{x_n}, x_1, \dots, x_{n-1}).
\end{aligned} \tag{A.8}$$

If $|1 - x_n| < 1$, we are in the region close to 1, and the analytic continuation now reads

$$\begin{aligned}
& F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\
&= \frac{\Gamma(c)\Gamma(c - a - b_n)}{\Gamma(c - a)\Gamma(c - b_n)} D_{0,n-1}^{(n)}(a, b_1, \dots, b_n; a + b_n - c + 1, c - b_n; \\
&\quad x_1, \dots, x_{n-1}, 1 - x_n) \\
&+ \frac{\Gamma(c)\Gamma(a + b_n - c)}{\Gamma(a)\Gamma(b_n)} (1 - x_n)^{c-a-b_n} D_{0,n-1}^{(n)}(c - b_n, b_1, \dots, b_{n-1}, c - a; c - a - b_n + 1, c - b_n; \\
&\quad x_1, \dots, x_{n-1}, 1 - x_n). \tag{A.9}
\end{aligned}$$

The first of the two continuations has the advantage that, in the region of large $|x_n|$, the first term on the right hand side is convergent even if all the other x_i , $i \neq n$, are close to 1. Thus, to find the analytic continuation in the case that one argument is large and another is close to 1, one only need to seek the analytic continuation of the second term on the right hand side. We will give here the result for the case that $|x_1|$ is large and $|x_n|$ is close to 1, since other cases can easily be obtained by permutations.

$$\begin{aligned}
& F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\
&= \frac{\Gamma(c)\Gamma(b_1 - a)}{\Gamma(b_1)\Gamma(c - a)} (-x_1)^{-a} F_D^{(n)}(a, 1 - c + a, b_2, \dots, b_n; 1 - b_1 + a; \frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}) \\
&+ \frac{\Gamma(c)}{\Gamma(a)} (-x_1)^{-b_1} \left(\frac{\Gamma(c - a - b_n)}{\Gamma(c - b_1 - b_n)} D_{1,2}^{(n)}(a - b_1, b_1, b_n, b_2, \dots, b_{n-1}; c - b_1 - b_n, a + b_n - c + 1; \right. \\
&\quad \left. \frac{1}{x_1}, 1 - x_n, x_2, \dots, x_{n-1}) \right. \\
&+ \left. \frac{\Gamma(a + b_n - c)}{\Gamma(b_n)} (1 - x_n)^{c-a-b_n} D_{1,2}^{(n)}(c - b_1 - b_n, b_1, c - a, b_2, \dots, b_{n-1}; c - b_1 - b_n, c - a - b_n + 1; \right. \\
&\quad \left. \frac{1}{x_1}, 1 - x_n, x_2, \dots, x_{n-1}) \right). \tag{A.10}
\end{aligned}$$

Unfortunately, these formulas are valid only for generic values of the parameters. In the cases relevant for the Seiberg–Witten periods, we have certain relations such as $a = b_i$ for some i which will cause singularities, if one attempts to analytically continue in the coordinate x_i . To obtain the correct answer, one has to take one further step, namely a limiting procedure. This is the well known Frobenius process, which essentially is nothing else than to consider the limit of $b_i = a + \epsilon$ for $\epsilon \rightarrow 0$. As an example, we present here one particular instance, namely

$$\begin{aligned}
& F_D^{(n)}(a, b_1, \dots, b_{n-1}, a; c; x_1, \dots, x_n) \\
&= \Gamma \left[\begin{matrix} c \\ a, c-a \end{matrix} \right] (-x_n)^{-a} \sum_M \sum_{m_n=0}^{\infty} \Gamma \left[\begin{matrix} c-a-|M| \\ c-a+|M| \end{matrix} \right] \frac{(a)_{|M|+m_n} (1-c+a)_{2|M|+m_n}}{(|M|+m_n)! m_n!} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} \times \\
&\quad \times (\log(-x_n) + h_{m_n}) \left(\frac{x_1}{x_n} \right)^{m_1} \dots \left(\frac{x_{n-1}}{x_n} \right)^{m_{n-1}} \left(\frac{1}{x_n} \right)^{m_n} \\
&\quad + \Gamma \left[\begin{matrix} c, c-a \\ a \end{matrix} \right] (-x_n)^{-a} \sum_M \sum_{m_n=0}^{|M|-1} \frac{(a)_{m_n} \Gamma(|M|-m_n)}{m_n! (c-a)_{|M|-m_n}} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} x_1^{m_1} \dots x_{n-1}^{m_{n-1}} \left(\frac{1}{x_n} \right)^{m_n},
\end{aligned} \tag{A.11}$$

where

$$h_{m_n} = \psi(1 + |M| + m_n) + \psi(1 + m_n) - \psi(a + |M| + m_n) - \psi(c - a - m_n), \tag{A.12}$$

and we have made use of multindex notation, so that $M = (m_1, \dots, m_{n-1})$, $|M| = \sum_{i=1}^{n-1} m_i$ and summation over M means summation over each m_i ($i = 1, \dots, n-1$) from 0 to ∞ .

This result can also be obtained ‘directly’ by using a suitable continuation formula for the Gaussian hypergeometric function.

Since such limiting procedures make the formulas extremely cumbersome, it is easier to work with the generic formulas, perform the necessary expansions with a computer algebra package and to then take the limit. The Lauricella functions which we encountered in our study of the $SU(2)$ case (cf. sect. 3.1) can be written in the following form: For $a(u)$, one has $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2})$, where $x = 1 - \frac{e_1}{e_4}$ with the notations used there, i.e. $e_1 = \sqrt{u - \Lambda^2}$ and $e_4 = \sqrt{u + \Lambda^2}$. For small x , this has a good power series expansion, namely,

$$F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2}) = 1 - \frac{3}{4}x + \frac{5}{32}x^2 + \frac{3}{123}x^3 - \frac{169}{262144}x^4 - \frac{1131}{1048576}x^5 + \dots \tag{A.13}$$

Now, again for small x , the dual period is proportional to $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, 2\frac{x-1}{x}, 2\frac{x-1}{x-2})$ which calls for an analytic continuation valid near the point $(0, \infty, 1)$. Actually, the first argument has modulus greater than 1, but since the Lauricella function is only polynomial in its first argument, we do not have to perform an analytic continuation for it. We find thus for the dual period

$$\begin{aligned}
& F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, -2, \frac{1}{2}; 1; 2\frac{x-1}{x}, 2, 2\frac{x-1}{x-2}) \\
&= -\frac{\sqrt{2x}}{\pi} \left(2 + 3x + \frac{27}{8}x^2 + \frac{19}{6}x^3 + \frac{9559}{4096}x^4 + \frac{12019}{12288}x^5 + \dots \right. \\
&\quad + \log(2) \left(3 + \frac{15}{4}x + \frac{135}{32}x^2 + \frac{561}{128}x^3 + \frac{989}{256}x^4 + \frac{11169}{4096}x^5 + \dots \right) \\
&\quad \left. - \log(x) \left(1 + \frac{5}{4}x + \frac{41}{32}x^2 + \frac{147}{128}x^3 + \frac{193}{256}x^4 + \frac{575}{4096}x^5 + \dots \right) \right). \tag{A.14}
\end{aligned}$$

Of course, we can also obtain $a_D(u)$ from $a(u)$ by a suitable analytic continuation. For this, one notes that the Lauricella functions are related to each other by (for the moment ignoring the fact that the three variables are all functions of $x = 1 - \frac{e_1}{e_4}$)

$$a(u) \propto F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, y, z\right), \quad (\text{A.15})$$

$$a_D(u) \propto F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; \frac{x}{z}, 1 - \frac{1}{y}, 1 - y\right), \quad (\text{A.16})$$

where the prefactors have been omitted. Such relations can be used, for example, to find the curve of marginal stability, where $a(u)/a_D(u)$ is real.

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